

# RIGIDITY SEQUENCES OF POWER RATIONALLY WEAKLY MIXING TRANSFORMATIONS

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**ABSTRACT.** We prove that a class of infinite measure preserving transformations, satisfying a "strong" weak mixing condition, generates all rigidity sequences of all conservative ergodic invertible measure preserving transformations defined on a Lebesgue  $\sigma$ -finite measure space.

## 1. INTRODUCTION

In the finite measure preservice setting, it is known that the weak mixing condition has many equivalent formulations. In the infinite measure preservice setting, many of these formulations lead to different families of infinite measure preserving transformations. For a general account of weak mixing conditions of infinite measure preserving transformations, please see [1, 3, 5, 10, 13, 16]. Figure 1 displays several distinct weak mixing conditions for infinite measure preserving transformations. The stronger weak mixing conditions appear higher in the diagram.

Most of the properties given in Figure 1 were defined previously by multiple authors. Many interesting results have been derived. In the finite measure preserving case, it was proven that the collection of weak mixing transformations generates all rigidity sequences for all ergodic transformations defined on a Lebesgue space. See [7] and [21] for details. For recent research on rigidity sequences in the  $\sigma$ -finite measure preserving case, see [10, 11, 17, 36, 37, 7]. Our primary goal is to give a class of infinite measure preserving, weak mixing transformations that generate all rigidity sequences of all ergodic finite measure preserving transformations. It was established in [36] and [37] that any rigidity sequence of a conservative ergodic infinite measure preserving transformation occurs as a rigidity sequence of a probability preserving weak mixing transformation. Thus, the class of infinite measure preserving transformations given here will generate

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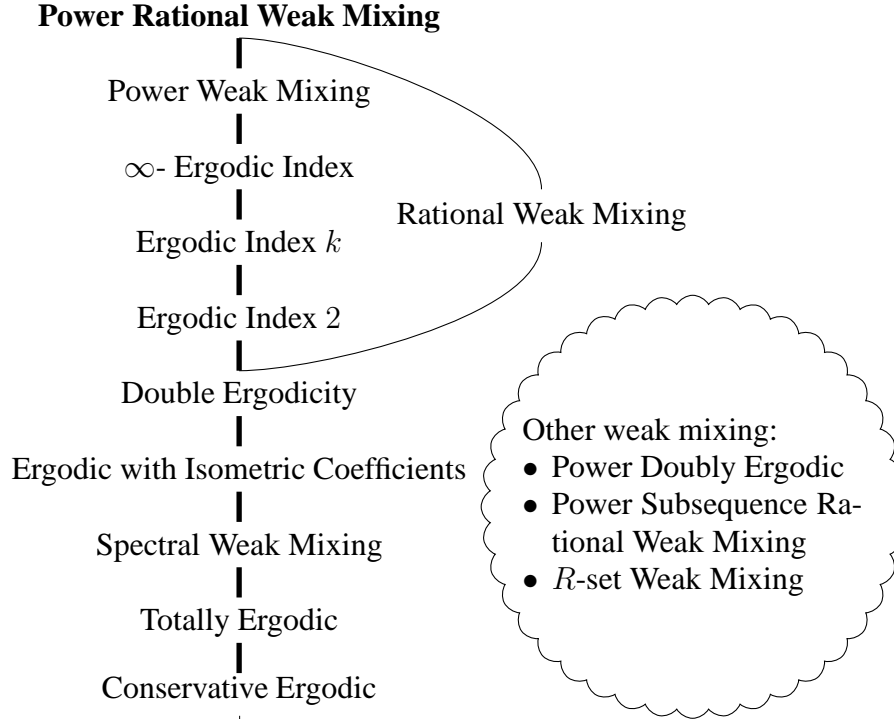


FIGURE 1. Weak Mixing with Infinite Measure

all rigidity sequences for all conservative ergodic  $\sigma$ -finite measure preserving transformations. We find this more interesting, if we are able to restrict the class to a collection that satisfies a strong form of weak mixing.

**Property 1.1** (Rational Weak Mixing). *For any set  $F \subset X$  of finite positive measure, define the intrinsic weight sequence of  $F$ ,  $u_k(F)$  and its accumulation by*

$$(1) \quad u_k(F) = \frac{\mu(F \cap T^k F)}{\mu(F)^2} \quad \text{and} \quad a_n(F) = \sum_{k=0}^{n-1} u_k(F \cap T^k F).$$

*A  $\sigma$ -finite measure preserving transformation is rationally weakly mixing, if it is conservative ergodic and there exists a set  $F$  of finite positive measure such that for all measurable sets  $A, B \subset F$ ,*

$$\lim_{n \rightarrow \infty} \frac{1}{a_n(F)} \sum_{k=0}^{n-1} |\mu(A \cap T^k B) - \mu(A)\mu(B)u_k(F)| = 0.$$

Rational weak mixing was first introduced by Aaronson [1] as a counterpart to mixing on a sequence of density one in the finite measure preserving case. It is known that rational weak mixing implies double ergodicity

which implies weak rational ergodicity [1], and subsequently, implies spectral weak mixing. Rational weak mixing does not imply ergodic index 2, in general.

In section 5, we prove the following result.

**Theorem 1.2.** *Let  $(X, \mathcal{B}, \mu)$  be a Lebesgue probability space. Suppose  $R$  is an invertible ergodic  $\mu$ -preserving transformation on  $(X, \mathcal{B}, \mu)$  with a rigidity sequence  $\rho_n \in \mathbb{N}$  for  $n \in \mathbb{N}$ . There exists an invertible infinite measure preserving transformation  $T$  that is both rigid on  $\rho_n$ , and is rationally weakly mixing.*

We use the tower multiplexing technique given in [7]. In that paper, a rigid weakly mixing transformation is produced from multiplexing an ergodic rigid transformation with a weakly mixing transformation. All transformations were finite measure preserving. Here we wish to tower multiplex a finite measure preserving transformation with an infinite measure preserving transformation. Let  $R$  be any finite measure preserving, ergodic transformation with rigidity sequence  $\rho_n$ . Since it was shown in [7] and [21] that any rigid sequence of an ergodic finite measure preserving transformation may be realized by a finite measure preserving weak mixing transformation, then it is sufficient to assume the starter transformation  $R$  is weak mixing, and rigid on  $\rho_n$ . The infinite measure preserving transformation  $S$  will be akin to the map  $S(x) = x + 1$  defined on  $[0, \infty)$ . The map  $S$  is invertible, but it's not ergodic, which is not required for this construction. We will produce an infinite measure preserving transformation  $T$  by multiplexing  $R$  with  $S$ :

$$T = \text{Mux}(\text{rigid weak mixing } R, \text{infinite measure preserving } S).$$

To strengthen our results, we introduce the notion of power rational weak mixing. An invertible infinite measure preserving transformation is *power rationally weakly mixing*, if given  $\ell \in \mathbb{N}$  and nonzero integers  $k_1, k_2, \dots, k_\ell$ , the Cartesian product

$$T^{k_1} \times T^{k_2} \times \dots \times T^{k_\ell}$$

is rationally weakly mixing. In this paper, assume all transformations are invertible and preserve a  $\sigma$ -finite measure defined on a Lebesgue space. Aaronson previously introduced the notion of power subsequence rational weak mixing [5], and power weak mixing was introduced in [16]. Note, power weak mixing is defined as a transformation where all finite Cartesian products of nonzero powers are ergodic. Thus, power rational weak mixing implies power weak mixing. In the final section, we extend Theorem 1.2 to

show that the class of power rational weak mixing, infinite measure preserving transformations generates all rigidity sequences for all finite measure preserving ergodic transformations.

**Theorem 1.3.** *Let  $(X, \mathcal{B}, \mu)$  be a Lebesgue probability space. Suppose  $R$  is an invertible ergodic  $\mu$ -preserving transformation on  $(X, \mathcal{B}, \mu)$  with a rigidity sequence  $\rho_n \in \mathbb{N}$  for  $n \in \mathbb{N}$ . There exists an invertible infinite measure preserving transformation  $T$  that is both rigid on  $\rho_n$ , and is power rationally weakly mixing.*

A result from [36], together with our result, show that rigidity sequences of ergodic finite measure preserving transformations coincide with rigidity sequences of conservative ergodic infinite measure preserving transformations. Moreover, the following corollary generalizes the main results from [7] and [21].

**Corollary 1.4.** *Let  $(X, \mathcal{B}, \mu)$  and  $(Y, \mathcal{A}, \nu)$  be Lebesgue  $\sigma$ -finite measure spaces. The set of rigidity sequences generated by all invertible power rationally weakly mixing transformations defined on  $(Y, \mathcal{A}, \nu)$  is identical to the set of rigidity sequences generated by all invertible conservative ergodic measure preserving transformations on  $(X, \mathcal{B}, \mu)$ .*

*Proof.* The case where  $\mu(X)$  and  $\nu(Y)$  are finite is handled in [7] and [21]. Suppose  $\mu(X) = \infty$  and  $\nu(Y) < \infty$ . It is proved in [36] that the set of rigidity sequences generated by all conservative ergodic measure preserving transformations on  $(X, \mathcal{B}, \mu)$  is contained in the set of rigidity sequences generated by all invertible weak mixing transformations defined on  $(Y, \mathcal{A}, \nu)$ . Theorem 1.3 shows these sets are equal. Likewise, the case where  $\mu(X) < \infty$  and  $\nu(Y) = \infty$  follows from Theorem 1.3 and [36]. Suppose both  $\mu(X)$  and  $\nu(Y)$  are infinite, and  $R$  is infinite measure preserving and conservative ergodic on  $(X, \mathcal{B}, \mu)$ . By [36], there exists a Poisson suspension  $R^*$  such that  $R^*$  is probability preserving, weak mixing and rigid on  $\rho_n$ . By Theorem 1.3, there exists an invertible infinite measure preserving power rationally weakly mixing  $T$  that is rigid on  $\rho_n$ .  $\square$

Note, recently, B. Fayad and A. Kanigowski [20] were able to construct a rigidity sequence for a finite measure preserving weak mixing transformation that is not rigid for any irrational rotation. This proves that the class of rigidity sequences for finite measure preserving weak mixing transformations is strictly larger than the class of rigidity sequences for finite measure preserving discrete spectrum transformations.

Also, recently, R. Bayless and K. Yancey [10] have given many explicit examples of infinite measure preserving transformations that are rigid and also satisfy a variety of weak mixing conditions (i.e. spectral weak mixing, rational ergodicity, ergodic Cartesian square).

## 2. INFINITE TOWERPLEX CONSTRUCTIONS

The towerplex method was first defined in section 2 of [7]. The use case here is simpler, since the only role of  $S$  is to supply  $T$  with infinite measure. There are a few main parameters that determine the final transformation. In [7], two sequences  $r_n$  and  $s_n$  are defined such that  $r_n$  represents the proportion of mass switching from the  $R$ -tower to the  $S$ -tower. Similarly,  $s_n$  represents the proportion of mass switching from the  $S$ -tower to the  $R$ -tower. Using the notation from [7], then the following values could be used to produce our desired transformation:

$$(2) \quad r_n = 0 \quad \text{and} \quad s_n = \frac{1}{n}.$$

Thus, for the constructions in this paper, we do not wish to transfer mass from the  $R$ -tower to the  $S$ -tower, and we wish to transfer measure  $1/n$  from the  $S$ -tower to the  $R$ -tower at stage  $n$ . It will be simpler to define  $S_n : Y_n \rightarrow Y_n$  such that

$$(3) \quad \mu(Y_n) = \frac{1}{n} \quad \text{and} \quad s_n = 1.$$

The transformation  $S_n : Y_n \rightarrow Y_n$  will be a cycle on  $h_n$  intervals, each with length  $1/nh_n$  for some  $h_n \in \mathbb{N}$ . The sequence  $h_n$  will correspond to heights of Rohklin towers for the transformation  $R_n$ . At stage  $n$  in the construction, a finite measure preserving, weak mixing transformation  $R_n : X_n \rightarrow X_n$  will be defined to be isomorphic to  $R$ . The set  $X_n$  will be specified inductively. Finally, we will specify a sequence of refining, generating partitions  $P_n$ . The assembled transformation  $T : \bigcup_{n=1}^{\infty} X_n \rightarrow \bigcup_{n=1}^{\infty} X_n$  will be invertible and ultimately (power) rationally weakly mixing with respect to  $\mu$ .

**2.1. Towerplex Chain.** Suppose  $h_n \in \mathbb{N}$  and  $\epsilon_n > 0$  are such that

$\sum_{n=1}^{\infty} 1/h_n < \infty$  and  $\sum_{n=1}^{\infty} \epsilon_n < \infty$ . Initialize  $R_1 = R$  on  $X = X_1$  (ex.  $X_1 = [0, 1)$ ). Let  $Y_1 = [1, 2)$  and define  $S_1(x) = x + 1/h_1$  on  $[1, 1 + (h_1 - 1)/h_1)$  and  $S(x) = x - (h_1 - 1)/h_1$  on  $[1 + (h_1 - 1)/h_1, 2)$ .

Let  $I_1, RI_1, \dots, R^{h_1-1}I_1$  be a Rohklin tower of height  $h_1$  such that  $\mu(E_1) < \epsilon_1$  where  $E_1 = X_1 \setminus \bigcup_{k=0}^{h_1-1} R^k I_1$ . Let  $X_2 = X_1 \cup Y_1$ , and  $d \in \mathbb{R}$  be such that

$$\frac{\mu(E_1) + d}{\mu(X_1) + \mu(Y_1) - d} = \frac{\mu(E_1) + d}{2 - d} = \frac{\mu(E_1)}{\mu(X_1)}.$$

Let  $J_1 = [1, 1 + 1/h_1)$  be the base of  $S_1$ . Let  $I_1^*$  be a subset of  $J_1$  with measure  $|d|/h_1$ . Let  $X'_1 = E_1 \cup \bigcup_{k=0}^{h_1-1} R^k I_1^*$ . Thus,  $I_1 \cup J_1 \setminus I_1^*, R_1 I_1 \cup S_1(J_1 \setminus I_1^*), \dots, R_1^{h_1-1} I_1 \cup S_1^{h_1-1}(J_1 \setminus I_1^*)$  are disjoint sets with equal measure. These

sets together with  $X'_1$  may form a rescaled Rohklin tower for  $R$ . Now, we specify how to define  $R_2$  consistently. Define

$$\mathcal{C}_1 = \left\{ \bigcap_{k=0}^{h_1-1} R_1^{-k} p_k : p_k \in P_1, p_k \subset R_1^k I_1 \right\}.$$

The collection  $\mathcal{C}_1$  generates a partition on  $I_n$ . Let  $P'_1$  be the smallest partition generated by the collection:

$$\bigvee_{k=0}^{h_1-1} \{R_1^k p : p \in \mathcal{C}_1\} \vee \{S_1^k(J_1 \setminus I_1^*)\}.$$

Define  $\tau_1 : X'_1 \rightarrow E_1$  as a measure preserving map between normalized spaces  $(X'_1, \mathbb{B} \cap X'_1, \frac{\mu}{\mu(X'_1)})$  and  $(E_1, \mathbb{B} \cap E_1, \frac{\mu}{\mu(E_1)})$ . Extend  $\tau_1$  to the new tower base,

$$\tau_1 : I_1 \cup J_1 \setminus I_1^* \rightarrow I_1$$

such that  $\tau_1$  preserves normalized measure between

$$\frac{\mu}{\mu(I_1 \cup J_1 \setminus I_1^*)} \text{ and } \frac{\mu}{\mu(I_1)}.$$

Define  $\tau_1$  on the remainder of the tower consistently such that

$$\tau_1(x) = \begin{cases} R_1^i \circ \tau_1 \circ R_1^{-i}(x) & \text{if } x \in R_1^i(I_1) \text{ for } 0 \leq i < h_1 \\ R_1^i \circ \tau_1 \circ S_1^{-i}(x) & \text{if } x \in S_1^i(J_1 \setminus I_1^*) \text{ for } 0 \leq i < h_1. \end{cases}$$

Since  $\tau_1$  is a contraction, we may require for all  $p \in P_1$ ,

$$\tau_1(p) \subset p.$$

Define  $R_2 : X_2 \rightarrow X_2$  as  $R_2 = \tau_1^{-1} \circ R_1 \circ \tau_1$ . Note

$$R_2(x) = \begin{cases} S_1(x) & \text{if } x \in S_1^i(J_1 \setminus I_1^*) \text{ for } 0 \leq i < h_1 - 1 \\ R_1(x) & \text{if } x \in R_1^i(I_1) \text{ for } 0 \leq i < h_1 - 1 \end{cases}$$

Clearly,  $R_2$  is isomorphic to  $R_1$  and  $R$ . Set  $Y_2 = [2, 2.5)$  and  $S_2 : Y_2 \rightarrow Y_2$  by  $S_2(x) = x + (1/2h_2)$  for  $x \in [2, 2.5 - 1/2h_2)$  and  $S_2(x) = x - (h_2 - 1)/2h_2$  for  $x \in [2.5 - 1/2h_2, 2.5)$ . Let  $b_1 = 2$  be the right endpoint of  $Y_1$  and let  $b_2 = 2.5$  be the right endpoint of  $Y_2$ .

**2.2. General Multiplexing Operation.** Let  $I_n, RI_n, \dots, R^{h_n-1}I_n$  be a Rohklin tower of height  $h_n$  such that  $\mu(E_n) < \epsilon_n$  where  $E_n = X_n \setminus \bigcup_{k=0}^{h_n-1} R^k I_n$ . Suppose  $b_n$  and  $Y_n = [b_{n-1}, b_n)$  have been defined. Let  $X_{n+1} = X_n \cup Y_n$ , and  $d_n \in \mathbb{R}$  be such that

$$\frac{\mu(E_n) + d_n}{\mu(X_n) + \mu(Y_n) - d_n} = \frac{\mu(E_n)}{\mu(X_n)}.$$

Let  $J_n = [0, 1/h_n)$  be the base of  $S_n$ . Let  $I_n^*$  be a subset of  $J_n$  with measure  $|d_n|/h_n$ . Let  $X'_n = E_n \cup \bigcup_{k=0}^{h_n-1} R^k I_n^*$ . The set  $X'_n \setminus E_n$  is the transfer set for

stage  $n$ . Thus,  $I_n \cup J_n \setminus I_n^*, R_n I_n \cup S_n(J_n \setminus I_n^*), \dots, R_n^{h_n-1} I_n \cup S_n^{h_n-1}(J_n \setminus I_n^*)$  are disjoint sets with equal measure. These sets together with  $X'_n$  may form a rescaled Rohklin tower for  $R$ . Now, we specify how to define  $R_{n+1}$  consistently. Define

$$\mathcal{C}_n = \left\{ \bigcap_{k=0}^{h_n-1} R_n^{-k} p_k : p_k \in P'_{n-1} \vee P_n, p_k \subset R_n^k I_n \right\}.$$

The collection  $\mathcal{C}_n$  generates a partition on  $I_n$ . Let  $P'_n$  be the smallest partition generated by the collection:

$$\bigvee_{k=0}^{h_n-1} \{R_n^k p : p \in \mathcal{C}_n\} \vee \{S_n^k(J_n \setminus I_n^*)\}.$$

Define  $\tau_n : X'_n \rightarrow E_n$  as a measure preserving map between normalized spaces  $(X'_n, \mathbb{B} \cap X'_n, \frac{\mu}{\mu(X'_n)})$  and  $(E_n, \mathbb{B} \cap E_n, \frac{\mu}{\mu(E_n)})$ . Extend  $\tau_n$  to the new tower base,

$$\tau_n : I_n \cup J_n \setminus I_n^* \rightarrow I_n$$

such that  $\tau_n$  preserves normalized measure between

$$\frac{\mu}{\mu(I_n \cup J_n \setminus I_n^*)} \text{ and } \frac{\mu}{\mu(I_n)}.$$

Define  $\tau_n$  on the remainder of the tower consistently such that

$$\tau_n(x) = \begin{cases} R_n^i \circ \tau_n \circ R_n^{-i}(x) & \text{if } x \in R_n^i(I_n) \text{ for } 0 \leq i < h_n \\ R_n^i \circ \tau_n \circ S_n^{-i}(x) & \text{if } x \in S_n^i(J_n \setminus I_n^*) \text{ for } 0 \leq i < h_n. \end{cases}$$

Since  $\tau_n$  is a contraction, we may require for all  $p \in P'_n$ ,

$$\tau_n(p) \subset p.$$

Define  $R_{n+1} : X_{n+1} \rightarrow X_{n+1}$  as  $R_{n+1} = \tau_n^{-1} \circ R_n \circ \tau_n$ . Note

$$R_{n+1}(x) = \begin{cases} S_n(x) & \text{if } x \in S_n^i(J_n \setminus I_n^*) \text{ for } 0 \leq i < h_n - 1 \\ R_n(x) & \text{if } x \in R_n^i(I_n) \text{ for } 0 \leq i < h_n - 1 \end{cases}$$

Clearly,  $R_{n+1}$  is isomorphic to  $R_n$  and  $R$ . Set  $b_{n+1} = b_n + 1/(n+1)$ ,  $Y_{n+1} = [b_n, b_{n+1})$  and transformation  $S_{n+1}$  similar to the previous stages. Also, let

$$Q_n = \{\tau_n(p) : p \in P'_n\}.$$

**2.3. The Limiting Transformation.** Define the transformation  $T_{n+1} : X_{n+1} \cup Y_{n+1} \rightarrow X_{n+1} \cup Y_{n+1}$  such that

$$T_{n+1}(x) = \begin{cases} R_{n+1}(x) & \text{if } x \in X_{n+1} \\ S_{n+1}(x) & \text{if } x \in Y_{n+1} \end{cases}$$

The set

$$D_n = \{x \in X_{n+1} : T_{n+1}(x) \neq T_n(x)\}$$

is determined by the top levels of the Rokhlin towers, the residual and the transfer set. Note the transfer set has measure  $d$ . Since this set is used to adjust the size of the residuals between stages, it can be bounded below a constant multiple of  $\epsilon_n$ . Thus, there is a fixed constant  $\kappa$ , independent of  $n$ , such that  $\mu(D_n) < \kappa(\epsilon_n + 1/h_n)$ . Since  $\sum_{n=1}^{\infty} (\epsilon_n + 1/h_n) < \infty$ ,  $T(x) = \lim_{n \rightarrow \infty} T_n(x)$  exists almost everywhere, and preserves Lebesgue measure. Without loss of generality, we may assume  $\kappa$  and  $h_n$  are chosen such that for  $n \in \mathbb{N}$ ,

$$\mu(D_n) < \kappa \epsilon_n.$$

Let  $X^+ = \bigcup_{n=1}^{\infty} X_n$ . Since

$$\mu(X^+) = \lim_{n \rightarrow \infty} (\mu(X_n) + \mu(Y_n)) = \infty,$$

then  $T$  is an invertible infinite measure preserving transformation. In the final section, we show there exist  $h_n$  and  $\epsilon_n$  such that  $T : X^+ \rightarrow X^+$  is power rationally weakly mixing.

### 3. ISOMORPHISM CHAIN CONSISTENCY

Suppose  $R$  is a weak mixing transformation on  $(X, \mathcal{B}, \mu)$  with rigidity sequence  $\rho_n$ . We will use the multiplexing procedure defined in the previous section to produce an invertible infinite measure preserving  $T$  such that  $T$  is rigid on  $\rho_n$  and (power) rationally weakly mixing. In the definition of rational weak mixing, let  $F = X_1 = X$  and assume without loss of generality that  $\mu(F) = 1$ . Let  $\mu_n$  be normalized Lebesgue probability measure on  $X_n$ . i.e.  $\mu_n = \mu/\mu(X_n)$ . Since each  $R_n$  is weakly mixing and finite measure preserving on  $X_n$ , then for all  $A, B \in P'_n$ ,

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=0}^{N-1} |\mu_n(A \cap R_n^i B) - \mu_n(A)\mu_n(B)| = 0.$$

If  $u_i(n) = \mu(F \cap R_n^i F)$  and  $a_N = \sum_{i=0}^{N-1} u_i(n)$ , then for each  $n \in \mathbb{N}$ ,

$$\lim_{N \rightarrow \infty} \frac{a_N \mu(X_n)}{N} = 1$$



and

$$\lim_{N \rightarrow \infty} \frac{1}{a_N} \sum_{i=0}^{N-1} |u_i(n) - \frac{1}{\mu(X_n)}| = 0.$$

This implies for all  $A, B \in P'_n$ ,

$$\lim_{N \rightarrow \infty} \frac{1}{a_N} \sum_{i=0}^{N-1} |\mu(A \cap R_n^i B) - \mu(A)\mu(B)u_i(n)| = 0.$$

Prior to establishing rational weak mixing, we prove a crucial lemma that was used in [7]. For  $p \in P'_n$ ,

$$\frac{\mu(p)}{\mu(\tau_n(p))} = \frac{\mu(X_{n+1})}{\mu(X_n)}.$$

It is straightforward to verify for any set  $A$  measurable relative to  $P'_n$ ,

$$\begin{aligned} \mu(A \triangle \tau_n A) &= \mu(A) - \mu(\tau_n(A)) \\ &\leq \mu(\tau_n A) \left[ \frac{\mu(X_{n+1})}{\mu(X_n)} - 1 \right] = \frac{\mu(\tau_n A)}{\mu(X_n)} [\mu(X_{n+1}) - \mu(X_n)]. \end{aligned}$$

and for any measurable set  $C \subset X_n$ ,

$$|\mu(\tau_n^{-1} C) - \mu(C)| < \left| \frac{\mu(X_{n+1})}{\mu(X_n)} - 1 \right|.$$

These two properties are used in the following lemma to show  $R_{n+1}$  inherits dynamical properties from  $R_n$  indefinitely over time.

**Lemma 3.1.** *Suppose  $\delta > 0$  and  $n \in \mathbb{N}$  is chosen such that*

$$\epsilon_n + \mu(Y_n) < \frac{\delta}{6}.$$

*Then for  $A, B \in Q_n$  and  $i \in \mathbb{N}$ , the following holds:*

- (1)  $|\mu(R_{n+1}^i A \cap B) - \mu(A)\mu(B)u_i(n+1)|$   
 $< |\mu(R_n^i A \cap B) - \mu(A)\mu(B)u_i(n)| + [\delta/\mu(X_n)];$
- (2)  $\mu(R_{n+1}^i A \triangle A) < \mu(R_n^i A \triangle A) + [\delta/2\mu(X_n)].$

*Proof.* For  $A, B \in Q_n$ , let  $A' = \tau_n^{-1} A$  and  $B' = \tau_n^{-1} B$ . Thus,  $\mu(A' \triangle A) = \mu(\tau_n^{-1}(A \setminus \tau_n A)) < \delta/6\mu(X_n)$  and  $\mu(B' \triangle B) < \frac{\delta}{6\mu(X_n)}$ . By applying the

triangle inequality several times, we get the following approximation:

$$\begin{aligned}
|\mu(R_{n+1}^i A \cap B) - \mu(R_n^i A \cap B)| & \\
&\leq |\mu(R_{n+1}^i A' \cap B') - \mu(R_n^i A \cap B)| + \frac{\delta}{3\mu(X_n)} \\
&= |\mu(\tau_n^{-1} R_n^i \tau_n A' \cap B') - \mu(R_n^i A \cap B)| + \frac{\delta}{3\mu(X_n)} \\
&= |\mu(\tau_n^{-1}(R_n^i \tau_n A' \cap \tau_n B')) - \mu(R_n^i A \cap B)| + \frac{\delta}{3\mu(X_n)} \\
&= |\mu(\tau_n^{-1}(R_n^i A \cap B)) - \mu(R_n^i A \cap B)| + \frac{\delta}{3\mu(X_n)} \\
&< \frac{\delta}{2\mu(X_n)}.
\end{aligned}$$

Similarly,

$$|\mu(R_{n+1}^i F \cap F) - \mu(R_n^i F \cap F)| < \frac{\delta}{2\mu(X_n)}.$$

Hence,

$$|\mu(R_{n+1}^i A \cap B) - \mu(A)\mu(B)u_i(n+1)| < |\mu(R_n^i A \cap B) - \mu(A)\mu(B)u_i(n)| + \frac{\delta}{\mu(X_n)}.$$

The second part of the lemma can be proven in a similar fashion using the triangle inequality.

$$\begin{aligned}
|\mu(R_{n+1}^i A \triangle A) - \mu(R_n^i A \triangle A)| &\leq |\mu(R_{n+1}^i A' \triangle A') - \mu(R_n^i A \triangle A)| + \frac{\delta}{3\mu(X_n)} \\
&= |\mu(\tau_n^{-1} R_n^i \tau_n A' \triangle A') - \mu(R_n^i A \triangle A)| + \frac{\delta}{3\mu(X_n)} \\
&= |\mu(\tau_n^{-1}(R_n^i A \triangle A)) - \mu(R_n^i A \triangle A)| + \frac{\delta}{3\mu(X_n)} \\
&< \frac{\delta}{2\mu(X_n)}
\end{aligned}$$

Therefore,

$$\mu(R_{n+1}^i A \triangle A) < \mu(R_n^i A \triangle A) + \frac{\delta}{2\mu(X_n)}$$

and our proof is complete.  $\square$

#### 4. APPROXIMATION

For probability preserving transformations, if the transformation is rigid on a dense collection of measurable sets, then the transformation is rigid on all measurable sets. Similarly, if a probability preserving transformation is

mixing on a fixed sequence for all measurable sets from a dense collection, then the transformation is mixing on the same sequence. Since the normalizing term  $a_N$  in the rationally weakly mixing condition may grow at a rate much slower than  $N$ , it is not clear this condition will hold for all measurable subsets  $A \subseteq F$ , when it holds for a dense collection of sets contained in  $F$ . In this section, we give conditions that allow extension of the rational weak mixing condition from a dense collection of sets in  $F$  to all measurable sets contained in  $F$ . Let  $P$  be a dense collection of sets, each a subset of  $F$ .

**Lemma 4.1.** *Suppose there exist a sequence of measurable sets  $F = X_1 \subset X_2 \subset \dots$ , and a sequence of natural numbers  $M_1 < M_2 < \dots$  such that for each  $A \in P$  and any sequence  $N_n$  satisfying  $M_n \leq N_n < M_{n+1}$ ,*

$$(4) \quad \lim_{n \rightarrow \infty} \frac{\mu(X_n)^2}{N_n} \sum_{i=0}^{N_n-1} |\mu(A \cap T^i A) - \mu(A)^2 \frac{1}{\mu(X_n)}| = 0.$$

Then for any measurable set  $E \subseteq F$  and  $A \in P$ ,

$$\lim_{n \rightarrow \infty} \frac{\mu(X_n)}{N_n} \sum_{i=0}^{N_n-1} |\mu(E \cap T^i A) - \mu(E)\mu(A) \frac{1}{\mu(X_n)}| = 0.$$

*Proof.* If not true, then there exists  $\delta > 0$  and  $\ell \in \mathbb{N}$  such that for  $n \geq \ell$ ,

$$\frac{\mu(X_n)}{N_n} \sum_{i=0}^{N_n-1} |\mu(E \cap T^i A) - \mu(E)\mu(A) \frac{1}{\mu(X_n)}| > 2\delta.$$

There exists  $\Gamma_n \subseteq \{0, 1, \dots, N_n - 1\}$  such that  $|\Gamma_n| \geq \delta N_n$ ,  $\mu(E \cap T^i A) - \mu(E)\mu(A)/\mu(X_n) \geq 0$  (or  $\leq 0$ ) for  $i \in \Gamma_n$  and

$$\frac{\mu(X_n)}{|\Gamma_n|} \sum_{i \in \Gamma_n} (\mu(E \cap T^i A) - \mu(E)\mu(A) \frac{1}{\mu(X_n)}) > \delta.$$

On the other hand, we can use the Cauchy-Schwarz inequality to obtain,

$$(5) \quad \frac{\mu(X_n)}{|\Gamma_n|} \sum_{i \in \Gamma_n} (\mu(E \cap T^i A) - \mu(E)\mu(A) \frac{1}{\mu(X_n)})$$

$$(6) \quad \leq \mu(X_n) \int_{X_n} \left( \frac{1}{|\Gamma_n|} \sum_{i \in \Gamma_n} I_{T^i A}(x) - \frac{\mu(A)}{\mu(X_n)} \right) I_E(x) d\mu$$

$$(7) \quad \leq \mu(X_n) \left[ \int_{X_n} \left( \frac{1}{|\Gamma_n|} \sum_{i \in \Gamma_n} I_{T^i A}(x) - \frac{\mu(A)}{\mu(X_n)} \right)^2 d\mu \right]^{\frac{1}{2}} \left[ \int_{X_n} I_E(x) d\mu \right]^{\frac{1}{2}}$$

$$(8) \quad \leq \left[ \frac{\mu(X_n)^2}{|\Gamma_n|^2} \sum_{i,j \in \Gamma_n} |\mu(T^i A \cap T^j A) - \frac{\mu(A)^2}{\mu(X_n)}| \right]^{\frac{1}{2}} \sqrt{\mu(E)}.$$

However, condition (4) implies that expression (8) converges to zero.  $\square$

The following lemma uses Lemma 4.1 to extend the rational weak mixing condition to all measurable subsets of  $F$ .

**Lemma 4.2.** *If  $T$  is conservative ergodic and satisfies the same conditions of Lemma 4.1, then  $T$  is rationally weakly mixing. In particular, for any measurable sets  $D, E \subseteq F$  and  $M_n \leq N_n < M_{n+1}$ ,*

$$\lim_{n \rightarrow \infty} \frac{\mu(X_n)}{N_n} \sum_{i=0}^{N_n-1} \left| \mu(E \cap T^i D) - \mu(E)\mu(D) \frac{1}{\mu(X_n)} \right| = 0.$$

*Proof.* Let  $D$  and  $E$  be measurable subsets of  $F$ , and let  $\eta > 0$ . Choose  $A, B \in \mathcal{P}$  such that  $\mu(A \triangle D) < \eta$  and  $\mu(B \triangle E) < \eta$ . Without loss of generality, let  $D = A \cap D$  and  $E = B \cap E$ . A straightforward application of the triangle inequality gives the following bounds,

$$\begin{aligned} (1) \quad & \frac{\mu(X_n)}{N_n} \sum_{i=0}^{N_n-1} \left| \mu(A)\mu(B) \frac{1}{\mu(X_n)} - \mu(A)\mu(E) \frac{1}{\mu(X_n)} \right| < \eta, \\ (2) \quad & \frac{\mu(X_n)}{N_n} \sum_{i=0}^{N_n-1} \left| \mu(A)\mu(B) \frac{1}{\mu(X_n)} - \mu(D)\mu(B) \frac{1}{\mu(X_n)} \right| < \eta, \\ (3) \quad & \frac{\mu(X_n)}{N_n} \sum_{i=0}^{N_n-1} \left| \mu(A)\mu(B) \frac{1}{\mu(X_n)} - \mu(D)\mu(E) \frac{1}{\mu(X_n)} \right| < 2\eta. \end{aligned}$$

From Lemma 4.1, we have that

$$\begin{aligned} (9) \quad & \frac{\mu(X_n)}{N_n} \sum_{i=0}^{N_n-1} |\mu(D \cap T^i B) - \mu(A \cap T^i B)| \\ (10) \quad & \leq \frac{\mu(X_n)}{N_n} \sum_{i=0}^{N_n-1} \left| \mu(D \cap T^i B) - \mu(D)\mu(B) \frac{1}{\mu(X_n)} \right| \\ (11) \quad & + \frac{\mu(X_n)}{N_n} \sum_{i=0}^{N_n-1} \left| \mu(A \cap T^i B) - \mu(A)\mu(B) \frac{1}{\mu(X_n)} \right| \\ (12) \quad & + \frac{\mu(X_n)}{N_n} \sum_{i=0}^{N_n-1} \left| \mu(A)\mu(B) \frac{1}{\mu(X_n)} - \mu(D)\mu(B) \frac{1}{\mu(X_n)} \right| \\ (13) \quad & \longrightarrow \eta_1 \leq \eta \end{aligned}$$

for some real number  $\eta_1 \geq 0$ . Similarly,

$$\frac{\mu(X_n)}{N_n} \sum_{i=0}^{N_n-1} |\mu(A \cap T^i E) - \mu(A \cap T^i B)| \rightarrow \eta_2$$

for some nonnegative real number  $\eta_2 \leq \eta$ . Finally, we have

$$(14) \quad \frac{\mu(X_n)}{N_n} \sum_{i=0}^{N_n-1} |\mu(D \cap T^i E) - \mu(D)\mu(E) \frac{1}{\mu(X_n)}|$$

$$(15) \quad \leq \frac{\mu(X_n)}{N_n} \sum_{i=0}^{N_n-1} |\mu(A \cap T^i B) - \mu(A)\mu(B) \frac{1}{\mu(X_n)}|$$

$$(16) \quad + \frac{\mu(X_n)}{N_n} \sum_{i=0}^{N_n-1} |\mu(A)\mu(B) \frac{1}{\mu(X_n)} - \mu(D)\mu(E) \frac{1}{\mu(X_n)}|$$

$$(17) \quad + \frac{\mu(X_n)}{N_n} \sum_{i=0}^{N_n-1} \left( \mu((A \setminus D) \cap T^i B) + \mu(A \cap T^i (B \setminus E)) \right)$$

$$(18) \quad \longrightarrow \eta_3 \leq 4\eta.$$

□

## 5. RATIONAL WEAK MIXING AND RIGID

To ensure conservativity and ergodicity, the same technique from [7] may be used, or directly modify the choice of  $M_n$ ,  $\epsilon_n$  and  $h_n$  below, to force  $F = X_1$  to sweep out. Suppose  $\delta_n > 0$  such that  $\lim_{n \rightarrow \infty} \delta_n = 0$ . Fix  $n \in \mathbb{N}$ . Suppose  $M_{n-1}$ ,  $h_{n-1}$  and  $\epsilon_{n-1}$  have been chosen. Choose  $M_n > \max \{h_{n-1}, M_{n-1}\}$  such that for all  $A \in P'_n$  and  $N \geq M_n$ ,

$$(19) \quad \frac{\mu(X_n)^2}{N} \sum_{i=0}^{N-1} |\mu(A \cap R_n^i A) - \mu(A)^2 \frac{1}{\mu(X_n)}| < \delta_n.$$

Choose  $\epsilon_n > 0$  and  $h_n > M_n$  such that

$$\epsilon_n n M_n < \epsilon_{n-1} \quad \text{and} \quad \frac{1}{h_n} n M_n < \epsilon_{n-1}.$$

*Proof of rational weak mixing.* Fix  $k \in \mathbb{N}$  and  $A \in P'_k$ . Suppose  $N_n \in \mathbb{N}$  such that  $M_n \leq N_n < M_{n+1}$ . By using the first approximation from Lemma 3.1,

$$\lim_{n \rightarrow \infty} \frac{\mu(X_{n+1})^2}{N_n} \sum_{i=0}^{N_n-1} |\mu(A \cap R_{n+1}^i A) - \mu(A)^2 \frac{1}{\mu(X_{n+1})}| = 0.$$

Set  $E_{n+1} = \{x \in X_n : T_{n+2}(x) \neq T_{n+1}(x)\}$ . Let

$$E'_{n+1} = \bigcup_{i=0}^{M_{n+1}-1} [T_{n+2}^{-i} E_{n+1} \cup T_{n+1}^{-i} E_{n+1}]$$

Thus,  $\mu(E'_{n+1}) < 2M_{n+1}\kappa\epsilon_{n+1}$ . For  $x \notin E'_{n+1}$ ,  $T_{n+2}^i(x) = T_{n+1}^i(x)$  for  $0 \leq i \leq M_{n+1}$ . Let  $E''_{n+1} = \bigcup_{k=n+1}^{\infty} E'_k$ . For  $x \notin E''_{n+1}$  and  $0 \leq i \leq M_{n+1}$ ,  $T^i(x) = T_{n+1}^i(x)$ . Also,

$$\mu(E''_{n+1}) < \sum_{k=n+1}^{\infty} 2M_k\kappa\epsilon_k < \frac{1}{n+1} \sum_{k=n+1}^{\infty} 2\kappa\epsilon_{k-1}$$

and  $\sum_{k=n+1}^{\infty} 2\kappa\epsilon_{k-1} \rightarrow 0$  as  $n \rightarrow \infty$ . Hence,

$$\lim_{n \rightarrow \infty} \frac{\mu(X_n)^2}{N_n} \sum_{i=0}^{N_n-1} |\mu(T^i A \cap A) - \mu(A)^2 \frac{1}{\mu(X_n)}| = 0.$$

Therefore, by Lemma 4.2,  $T$  is rationally weakly mixing.  $\square$

Rigidity on  $\rho_n$  can be established in a similar fashion, using approximation (2) from Lemma 3.1, and similar choices for  $M_n$ ,  $\epsilon_n$  and  $h_n$ .

## 6. POWER RATIONAL WEAK MIXING

We show that the techniques given in this paper can be applied to the class of power rational weak mixing transformations. We can use the constructions defined previously in this paper. We need to update the choice of the parameters  $M_n$ ,  $\epsilon_n$  and  $h_n$ . Let  $V$  be the collection of all finite vectors comprised of nonzero integers. The collection  $V$  is countable, so we can order  $V = \{v_1, v_2, \dots\}$ . For each  $v \in V$  and  $n \in \mathbb{N}$ , define the finite measure preserving transformation

$$\mathcal{R}_{n,v} = R_n^{u_1} \times R_n^{u_2} \times \dots \times R_n^{u_{|v|}}$$

where  $v = \langle u_1, u_2, \dots, u_{|v|} \rangle$ . Products of sets from  $P'_n$  may be used to produce a finite approximating collection for the Cartesian product space. Also, Lemma 3.1 may be extended in a straightforward manner to subsets of the product space. Note the map  $\tau_n$  can be applied pointwise to produce an analogous isomorphism on products. Suppose  $j_n$  is a sequence of natural numbers such that  $\lim_{n \rightarrow \infty} j_n = \infty$ . Now, replace condition (19) above with the following condition,

$$\frac{\mu(X_n)^2}{N} \sum_{i=0}^{N-1} |\mu(A \cap \mathcal{R}_{n,v_j}^i A) - \mu(A)^2 \frac{1}{\mu(X_n)}| < \delta_n$$

and require this hold for  $1 \leq j \leq j_n$  and  $N \geq M_n$ . This is possible, since  $R_n$  is finite measure preserving, weak mixing, and all finite Cartesian products of nonzero powers of  $R_n$  will be weak mixing. In a manner similar to the case of a single transformation  $T$ , we can force the product transformation to be conservative ergodic by ensuring the set  $X_1 \times X_1 \times \dots \times X_1$  sweeps out under the product transformation. The rest of the arguments

from the proof of Theorem 1.2 go through in the same manner, but with  $\mathcal{R}_{n,v_j}$  replacing  $R_n$  and

$$T^{u_1} \times T^{u_2} \times \dots \times T^{u_{|v_j|}}$$

replacing  $T$ . If the sequence  $j_n$  grows slowly enough, then we still have  $\sum_{k=n+1}^{\infty} 2\kappa\epsilon_{k-1} \rightarrow 0$  as  $n \rightarrow \infty$ , and our result follows.  $\square$

The corollaries below follow from Theorem 1.3 and corollaries given in [7]. Given a sequence  $\mathcal{A}$ , define the density function  $g_{\mathcal{A}} : \mathbb{N} \rightarrow [0, 1]$  such that  $g_{\mathcal{A}}(k) = \#(\mathcal{A} \cap \{1, 2, \dots, k\})/k$ .

**Corollary 6.1.** *Given any real-valued function  $f : \mathbb{N} \rightarrow (0, \infty)$  such that*

$$\lim_{n \rightarrow \infty} f(n) = 0,$$

*there exists an infinite measure preserving, power rationally weakly mixing transformation with rigidity sequence  $\mathcal{A}$  such that*

$$\lim_{n \rightarrow \infty} \frac{f(n)}{g_{\mathcal{A}}(n)} = 0.$$

*Also, there exist infinite measure preserving, power rationally weakly mixing transformations with rigidity sequences  $\rho_n$  satisfying*

$$\lim_{n \rightarrow \infty} \frac{\rho_{n+1}}{\rho_n} = 1.$$

**Corollary 6.2.** *Let  $\alpha \in (0, 1)$  be any irrational number, and let  $\rho_n$  be a sequence of natural numbers satisfying*

$$\lim_{n \rightarrow \infty} |\exp(2\pi i \alpha \rho_n) - 1| = 0.$$

*Then there exists an infinite measure preserving, power rationally weakly mixing transformation  $T$  such that  $\rho_n$  is a rigidity sequence for  $T$ .*

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